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Integration of Smooth Functions and φ-Discrepancy

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1. INTRODUCTION

A sequence $p_1, p_2, ..., p_N$ of nonnegative real numbers is said to be a weight sequence if

$$\sum_{k=1}^{N} p_k = 1.$$

Suppose we are given a sequence $x_1, x_2, ..., x_N$ of numbers in the unit interval $[0, 1]$ and a weight sequence $p_1, p_2, ..., p_N$. We recall that the discrepancy D_N of the sequence $x_1, x_2, ..., x_N$ with respect to the weight sequence $p_1, p_2, ..., p_N$ is defined as

$$D_N = \sup_{0 \leqslant x \leqslant 1} |g(x)|,$$

where

$$g(x) = x - \sum_{\substack{1 \leqslant k \leqslant N \\ x_k < x}} p_k.$$

In [1] we introduced the following notion of φ-discrepancy.

Let φ: $[0, 1] \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) φ is nondecreasing on $[0, 1]$,

(ii) $\lim_{x \to 0+} \varphi(x) = 0$,

(iii) $\varphi(x) > 0$ for $x > 0$.

The the number

$$D_N^{(\varphi)} = \int_0^1 \varphi(|g(x)|)\, dx$$

is said to be the φ-discrepancy of the sequence $x_1, x_2, ..., x_N$ with respect to the weight sequence $p_1, p_2, ..., p_N$.

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Obviously,

$$D_N^{(\varphi)} \leqslant \varphi(D_N). \tag{1}$$

In [2] we have proved that if the numbers $x_1, x_2, ..., x_N$ are ordered according to their magnitude, i.e.,

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1, \tag{2}$$

then

$$D_N^{(\varphi)} = \int_0^1 \varphi(h(x))\, dx, \tag{3}$$

where the function h is defined on $[0, 1]$ by

$$h(x) = |x - x_k| \qquad \text{if} \quad x \in [a_{k-1}, a_k),\ 1 \leqslant k \leqslant N. \tag{4}$$

Here and throughout the numbers $a_0, a_1, ..., a_N$ are given by

$$a_0 = 0, \qquad a_k = \sum_{i=1}^{k} p_i \qquad (k = 1, 2, ..., N).$$

(We also assume that $[a, b) = [a, b]$ if $b = 1$.)

We shall consider in this paper quadrature formulae of the type

$$\int_0^1 f(x)\, dx = \sum_{k=1}^{N} \sum_{j=0}^{r} A_{kj} f^{(j)}(x_k) + R_N^{(r)}(f), \tag{5}$$

where the function f is r-times differentiable on $[0, 1]$ and coefficients A_{kj} are given by

$$A_{kj} = \frac{(a_k - x_k)^{j+1} - (a_{k-1} - x_k)^{j+1}}{(j+1)!}.$$

For the history of quadrature formulae of this type see [3]. Note that in the case $r = 0$, the formula (5) can be written in the form

$$\int_0^1 f(x)\, dx = \sum_{k=1}^{N} p_k f(x_k) + R_N^{(0)}(f). \tag{6}$$

Let us recall that a continuous function ω defined on $[0, \infty)$ is called a modulus of continuity if $\omega(0) = 0$ and

$$0 \leqslant \omega(y) - \omega(x) \leqslant \omega(y - x) \qquad \text{for} \quad 0 \leqslant x \leqslant y.$$

For a nonnegative integer r and a modulus of continuity ω, we denote by W^rH^ω the set of all functions f defined on $[0, 1]$ for which the inequality

$$|f^{(r)}(x)-f^{(r)}(y)| \leqslant \omega(|x-y|)$$

holds for all x and y belonging to $[0, 1]$. We shall write in what follows $W^rH_\alpha(C)$ instead of W^rH^ω if $\omega(t)=Ct^\alpha$, where $C>0$ and $0<\alpha\leqslant 1$.

In this paper, we investigate the error of the quadrature formulae of the type (5) in the classes W^rH^ω. An exact (in a certain sense) estimate for the integration error is obtained by means of the φ-discrepancy of the sequence $x_1, x_2, ..., x_N$ with respect to the weight sequence $p_1, p_2, ..., p_N$.

2. Statement of the Main Result

We shall suppose in what follows that the condition (2) holds.

THEOREM 1. *Let r be a nonnegative integer and ω be a modulus of continuity. Then*

$$\sup_{f \in W^rH^\omega} |R_N^{(r)}(f)| \leqslant D_N^{(\varphi)}, \tag{7}$$

where the function φ is given by

$$\varphi(x)=\begin{cases} \omega(x) & \textit{if} \quad r=0, \\ \dfrac{x^r}{(r-1)!}\displaystyle\int_0^1 (1-t)^{r-1}\omega(tx)\,dt & \textit{if} \quad r\geqslant 1. \end{cases} \tag{8}$$

Moreover, the inequality (7) *changes into equality if either*

$$\omega(t)=Ct \tag{9}$$

(*where C is an absolute constant*) *or r is an even integer and the numbers x_k and p_k are related by*

$$a_k=\frac{x_k+x_{k+1}}{2} \qquad (k=1, 2, ..., N-1). \tag{10}$$

Taking into account (1), we immediately obtain from Theorem 1 the following

COROLLARY 1. *Let r be a nonnegative integer and ω be a modulus of continuity. Then*

$$\sup_{f \in W^rH^\omega} |R_N^{(r)}(f)| \leqslant \varphi(D_N), \tag{11}$$

where the function φ is given by (8).

In the case $r=0$ and $p_1=p_2=\cdots=p_N=1/N$, the estimate (11) was proved by Niederreiter [4]. For arbitrary weights (also in the case $r=0$) this estimates was obtained by the author [5] and rediscovered by Niederreiter and Tichy [6]. In the case $r\geqslant 1$ it was proved by the author and Kirov [7].

It is easy to check that if (9) holds then (8) can be written in the form

$$\varphi(x)=\frac{Cx^{r+1}}{(r+1)!}. \tag{12}$$

Therefore, from Theorem 1 we get the following

COROLLARY 2. *Let r be a nonnegative integer. Then*

$$\sup_{f\in W^r H_1(C)} |R_N^{(r)}(f)|=\frac{C}{(r+1)!}\int_0^1 |g(x)|^{r+1}\,dx.$$

This estimate in the case $r=0$ was proved by Sobol' [8]. In the case $r\geqslant 1$ it was proved by the author [3]. We would like to note that some other results of [3] are consequences of Theorem 1 as well.

Remark. It is not difficult to see that in the case $r=0$, the requirement (2) is unnecessary (in Theorem 1 and both corollaries).

5. APPROXIMATION OF FUNCTIONS AND φ-DISCREPANCY

Suppose that a function f is r-times differentiable on $[0,1]$. Then for every $x\in[0,1]$, determine an integer k $(0\leqslant k\leqslant N)$ such that $x\in[a_{k-1},a_k)$ and define $L_N^{(r)}(f;x)$ by

$$L_N^{(r)}(f;x)=\sum_{j=0}^{r}\frac{f^{(j)}(x_k)}{j!}(x-x_k)^j.$$

Approximation properties of the linear operators $L_N^{(r)}$ $(r=0,1,2,...)$ in the L^p metric were studied by several authors (see [3]). In this section, we continue these investigations but in more general metrics.

Let Φ be a nondecreasing function on $[0,\infty)$. Denote for simplicity,

$$\|F\|_\Phi=\int_0^1 \Phi(|F(x)|)\,dx,$$

where F is a function, defined on $[0,1]$. (For example, one can assume that $\|\cdot\|_\Phi$ is a norm of a functional space L^Φ [9].) In the next theorem, we obtain an exact (in a certain sense) upper estimate for the approximation error $\|f - L_N^{(r)}(f)\|_\Phi$ in the classes W^rH^ω.

THEOREM 2. *Let r be a nonnegative integer and ω be a modulus of continuity. Then*

$$\sup_{f \in W^rH^\omega} \|f - L_N^{(r)}(f)\|_\Phi \leqslant D_N^{(\psi)}, \tag{13}$$

where $\psi(x) = \Phi(\varphi(x))$ and the function φ is given by (8). *Moreover, the inequality* (13) *changes into equality if either* (9) *or* (10) *holds.*

Proof. Suppose $f \in W^rH^\omega$. First, we consider the case $r \geqslant 1$. It can be proved [7] that for each $x \in [a_{k-1}, a_k)$, $1 \leqslant k \leqslant N$, we have

$$f(x) - L_N^{(r)}(f; x) = \frac{(x - x_k)^r}{(r-1)!} \int_0^1 (1-t)^{r-1} [f^{(r)}(x_k + (x - x_k)\, t) - f^{(r)}(x_k)]\, dt. \tag{14}$$

Therefore,

$$\begin{aligned} \|f - L_N^{(r)}(f)\|_\Phi &= \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} \Phi(|f(x) - L_N^{(r)}(f; x)|)\, dx \\ &= \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} \Phi\left(\frac{|x - x_k|^r}{(r-1)!} \left| \int_0^1 (1-t)^{r-1} [f^{(r)}(x_k + (x - x_k)\, t) \right.\right. \\ &\qquad \left.\left. - f^{(r)}(x_k)]\, dt \right|\right) dx. \end{aligned} \tag{15}$$

From this we deduce

$$\begin{aligned} \|f - L_N^{(r)}(f)\|_\Phi &\leqslant \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} \Phi\left(\frac{|x - x_k|^r}{(r-1)!} \int_0^1 (1-t)^{r-1} \omega(|x - x_k|\, t)\, dt\right) dx \\ &= \int_0^1 \Phi\left(\frac{h(x)^r}{(r-1)!} \int_0^1 (1-t)^{r-1} \omega(th(x))\, dt\right) dx \\ &= \int_0^1 \Phi(\varphi(h(x)))\, dx = \int_0^1 \psi(h(x))\, dx, \end{aligned} \tag{16}$$

where the function h is defined on $[0, 1]$ by (4). Applying (3), we obtain from (16) that

$$\|f - L_N^{(r)}(f)\|_\Phi \leqslant D_N^{(\psi)}. \tag{17}$$

Now, let us consider the case $r = 0$. Obviously,

$$\|f - L_N^{(0)}(f)\|_\Phi = \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} \Phi(|f(x) - f(x_k)|)\, dx. \tag{18}$$

Consequently,

$$\|f - L_N^{(0)}(f)\|_\Phi \leqslant \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} \Phi(\omega(|x - x_k|))\, dx$$
$$= \int_0^1 \Phi(\omega(h(x)))\, dx = \int_0^1 \psi(h(x))\, dx.$$

From this and (3) we again obtain (17) and so (13) is proved. Now we shall prove the second part of the theorem.

Let (9) hold. Then the function φ is given by (12). Define the function $\bar{f}$ on $[0, 1]$ by

$$\bar{f}^{(r)}(x) = Cx. \tag{19}$$

Evidently,

$$\bar{f} \in W^r H_1(C) = W^r H^\omega.$$

From (15), (18), and (3), we deduce

$$\|\bar{f} - L_N^{(r)}(\bar{f})\|_\Phi = \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} \Phi\left(\frac{|x - x_k|^{r+1}}{(r+1)!}\right) dx$$
$$= \int_0^1 \Phi\left(\frac{h(x)^{r+1}}{(r+1)!}\right) dx = \int_0^1 \Phi(\varphi(h(x)))\, dx$$
$$= \int_0^1 \psi(h(x))\, dx = D_N^{(\psi)}.$$

Now, let (10) hold. Define the function $\bar{f}$ on $[0, 1]$ by

$$\bar{f}^{(r)}(x) = \omega(h(x)). \tag{20}$$

Obviously,

$$\bar{f} \in W^r H^\omega.$$

From (15), (18), and (3), we deduce

$$\|\tilde{f}-L_N^{(r)}(\tilde{f})\|_\Phi=\int_0^1 \Phi(\varphi(h(x)))\,dx$$

$$=\int_0^1 \psi(h(x))\,dx=D_N^{(\psi)}.$$

Thus, Theorem 2 is proved.

Setting $\Phi(x)=x$ $(0<p<\infty)$ in Theorem 2, we get the following

COROLLARY 3. *Suppose* $0<p<\infty$. *Let* r *be a nonnegative integer and* ω *be a modulus of continuity. Then*

$$\sup_{f\in W^rH^\omega}\|f-L_N^{(r)}(f)\|_{L^p}\leqslant\left(\int_0^1 \varphi(|g(x)|)^p\,dx\right)^{1/p}, \tag{21}$$

where the function φ *is given by* (8) *and*

$$\|F\|_{L^p}=\left(\int_0^1 |F(x)|^p\,dx\right)^{1/p}.$$

Moreover, the inequality (21) *changes into equality if either* (9) *or* (10) *holds.*

Passing to the limit as $p\to\infty$ in Corollary 3, we get the following

COROLLARY 4. *Let* r *be a nonnegative integer and* ω *be a modulus of continuity. Then*

$$\sup_{f\in W^rH^\omega}\|f-L_N^{(r)}(f)\|_C\leqslant\varphi(D_N), \tag{22}$$

where the function φ *is given by* (8) *and*

$$\|F\|_C=\sup_{0\leqslant x\leqslant 1}|F(x)|.$$

Moreover, the inequality (22) *changes into equality if either* (9) *or* (10) *holds.*

The first part of this corollary was proved in [7].

4. Proof of Theorem 1

It is easy to check that

$$R_N^{(r)}(f)=\int_0^1 [f(x)-L_N^{(r)}(f;x)]\,dx. \tag{23}$$

Therefore,

$$|R_N^{(r)}(f)| \leqslant \|f - L_N^{(r)}(f)\|_L. \tag{24}$$

Now the estimate (7) follows from (24) and Corollary 3 ($p=1$). We begin the proof of the second part of the theorem.

Obviously,

$$R_N^{(0)}(f) = \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} [f(x) - f(x_k)]\, dx. \tag{25}$$

For the case $r \geqslant 1$, it follows from (23) and (14) that

$$R_N^{(r)}(f) = \frac{1}{(r-1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x - x_k)^r \times \left(\int_0^1 (1-t)^{r-1} [f^{(r)}(x_k + (x - x_k)\, t) - f^{(r)}(x_k)]\, dt \right) dx. \tag{26}$$

Let (9) hold and r be an odd integer. From (25), (26), and (3), we obtain

$$R_N^{(r)}(\tilde{f}) = \frac{C}{(r+1)!} \int_0^1 h(x)^{r+1}\, dx = \int_0^1 \varphi(h(x))\, dx = D_N^{(\varphi)},$$

where the function $\tilde{f}$ is defined on $[0, 1]$ by (19).

Now let (9) hold and r be an even integer. Define the function $\tilde{h}$ on $[0, 1]$ by

$$\tilde{h}(x) = C_k + C|x - x_k| \qquad \text{if} \quad x \in [a_{k-1}, a_k),\ 1 \leqslant k \leqslant N,$$

where the numbers $C_1, C_2, ..., C_N$ are given by

$$C_0 = 0, \qquad C_{k+1} = C_k + C|a_k - x_k| - C|a_k - x_{k+1}|, \qquad k = 1, 2, ..., N.$$

Then define the function $\tilde{f}$ on $[0, 1]$ by

$$\tilde{f}^{(r)}(x) = \omega(\tilde{h}(x)).$$

In [3] we have proved that

$$\tilde{h} \in W^0 H_1(C).$$

Hence,

$$\tilde{f} \in W^r H_1(C) = W^r H^\omega.$$

From (25), (26), and (3), we obtain

$$R_N^{(r)}(\bar{f}) = \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x - x_k)^r |x - x_k| \, dx$$
$$= \frac{C}{(r+1)!} \int_0^1 h(x)^{r+1} dx = \int_0^1 \varphi(h(x)) \, dx = D_N^{(\varphi)}.$$

Finally, let (10) hold and r be an even integer. Then from (25), (26), and (3), we get

$$R_N^{(r)}(\bar{f}) = \int_0^1 \varphi(h(x)) \, dx = D_N^{(\varphi)},$$

where the function $\bar{f}$ is defined on [0, 1] by (20). Thus, Theorem 1 is proved.

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